Relationships among Probability Distributions – Prof. Richard B. Goldstein

Gamma Function

[Formulas from Abramowitz & Stegun – Handbook of Mathematical Functions]

$$\Gamma(a) = \int_0^\infty t^{a-1} e^{-t}\, dt \qquad P(a,x) = \frac{1}{\Gamma(a)} \int_0^x t^{a-1} e^{-t}\, dt$$ (a.k.a. Incomplete Gamma Function)

Chi-Square: $P(\chi^2 \mid \upsilon) = P(a,x)$ where $\upsilon = 2a$ and $\chi^2 = 2x$

Poisson distribution: $Q(\chi^2 \mid \upsilon) = \sum_{j=0}^{c-1} e^{-m} \frac{m^j}{j!}$ where $c = \frac{\upsilon}{2},\ m = \frac{\chi^2}{2}$ (υ even)

$$Q(\chi^2 \mid \upsilon) - Q(\chi^2 \mid \upsilon - 2) = e^{-m} \frac{m^{c-1}}{(c-1)!}$$

Beta Function

$B(x,y) = B(y,x) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \frac{(x-1)!(y-1)!}{(x+y-1)!}$ where the last expression is for integer x and y

Incomplete Beta Function: $I_x(a,b) = \frac{B(x;a,b)}{B(a,b)}$ where $B(x;a,b) = \int_0^x t^{a-1}(1-t)^{b-1}\, dt$

Binomial distribution: $\sum_{k=a}^{n} \binom{n}{k} p^k (1-p)^{n-k} = I_p(a, n-a+1)$

$$\binom{n}{k} p^k (1-p)^{n-k} = I_p(k, n-k+1) - I_p(k+1, n-k)$$

Student-t distribution: $\frac{1}{2}[1 - A(t \mid \upsilon)] = \frac{1}{2} I_x\left(\frac{\upsilon}{2}, \frac{1}{2}\right)$ where $A(t \mid \upsilon) = P\{|T| \le t\} = \int_{-t}^{t} f(x)\, dx$

Fisher-Snedecor F: $Q(F \mid \upsilon_1, \upsilon_2) = I_x\left(\frac{\upsilon_2}{2}, \frac{\upsilon_1}{2}\right) \quad x = \frac{\upsilon_2}{\upsilon_2 + \upsilon_1 F}$

F to Student-t: $Q(F \mid \upsilon_1 = 1, \upsilon_2) = 1 - A(t \mid \upsilon_2) \quad t = \sqrt{F}$

Johnson System (transforms to the normal)

Lognormal: $z = \ln(x)/\sigma$

Johnson-SB: $z = \alpha_1 + \alpha_2 \ln(x/(1-x))$

Johnson-SU $z = \alpha_1 + \alpha_2 \sinh^{-1}(x) = \alpha_1 + \alpha_2 \ln(x + \sqrt{x^2 + 1})$