Johnson System of Distributions – Statistics – Prof. Richard B. Goldstein

Transformation	Function	Common Name and Domain
$z = \ln(x)$	$f(x) = \dfrac{e^{-\frac{(\ln x)^2}{2\sigma^2}}}{x\sigma\sqrt{2\pi}}$	Lognormal S_L $[0, \infty)$
$z = \ln\left(\dfrac{x}{1-x}\right)$	$f(x) = \dfrac{\alpha_2}{x(1-x)\sqrt{2\pi}} e^{-\frac{\left[\alpha_1+\alpha_2\ln\left(\frac{x}{1-x}\right)\right]^2}{2}}$	Johnson S_B $[0, 1]$
$z = \ln\left(x+\sqrt{x^2+1}\right)$ $z = \sinh^{-1}(x)$	$f(x) = \dfrac{\alpha_2}{\sqrt{2\pi}\sqrt{x^2+1}} e^{-\frac{\left[\alpha_1+\alpha_2\ln\left(x+\sqrt{x^2+1}\right)\right]^2}{2}}$	Johnson S_U $(-\infty,\infty)$

Lognormal (transition between SB and SU below)

$$f(x,\theta,m,\sigma) = \frac{e^{-\frac{\left[\ln\left(\frac{x-\theta}{m}\right)\right]^2}{2\sigma^2}}}{(x-\theta)\sigma\sqrt{2\pi}}$$

$$F(x) = \Phi\left(\frac{\ln\left(\frac{x-\theta}{m}\right)}{\sigma}\right) \quad or \quad t = \theta + me^{\sigma\Phi^{-1}(P)} \quad for\ P^{th}\ percentile$$

In Microsoft Excel©

F(x) is found using NORMDIST(LN((x-θ)/m), 0, σ, 1)

t is found using θ+m*EXP(σ*NORMINV(P, 0, 1))

Johnson SB

$$f(x,\alpha_1,\alpha_2,p,q) = \frac{\alpha_2(q-p)}{(x-p)(q-x)}\phi\left(\alpha_1+\alpha_2\ln\left(\frac{x-p}{q-x}\right)\right) \text{for } p<x<q,\ \alpha_2>0$$

$$F(x) = \Phi\left(\alpha_1+\alpha_2\ln\left(\frac{x-p}{q-x}\right)\right) \quad \text{or} \quad t = \frac{q\exp\left(\frac{\Phi^{-1}(P)-\alpha_1}{\alpha_2}\right)+p}{\exp\left(\frac{\Phi^{-1}(P)-\alpha_1}{\alpha_2}\right)+1} \quad \text{for } P^{th} \text{ percentile}$$

In Microsoft Excel©

F(x) is found using NORMDIST(α_1+ α_2LN((x-p)/(q-x)),0,1,1)

t is found using cell#1: EXP((NORMINV(P, 0, 1)-α_1)/α_2)

cell#2: (q*cell#1+p)/(cell#1+1)

Johnson SU

$$f(x,\alpha_1,\alpha_2,\mu,\sigma) = \frac{\alpha_2}{\sigma\sqrt{\left(\frac{x-\mu}{\sigma}\right)^2+1}}\,\phi\left(\alpha_1+\alpha_2\ln\left(\frac{x-\mu}{\sigma}+\sqrt{\left(\frac{x-\mu}{\sigma}\right)^2+1}\right)\right) \text{ for } \alpha_2 > 0$$

note: $\sinh^{-1}(x) = \ln\left(x+\sqrt{x^2+1}\right)$ *where* $\sinh(x) = \frac{e^x - e^{-x}}{2}$

$F(x) = \Phi\left(\alpha_1+\alpha_2\sinh^{-1}\left(\frac{x-\mu}{\sigma}\right)\right)$ or $t = \mu+\sigma\sinh\left(\frac{\Phi^{-1}(P)-\alpha_1}{\alpha_2}\right)$ for P^{th} percentile

In Microsoft Excel©

F(x) is found using NORMDIST(α_1+ α_2ASINH((x-μ)/σ),0,1,1)

t is found using μ+σ*SINH((NORMINV(P, 0, 1)-α_1)/α_2)

The (γ_1, γ_2) plane for Johnson distributions