Functions of a Random Variable & Moment Generating Functions – Prof. Richard B. Goldstein

Discrete

$Y = u(X)$ is a one-to-one (1-1) transform

Let $y = u(x)$ and the inverse $x = w(y)$. Then $g(y) = f[w(y)]$ is the probability distribution of Y

$Y_1 = u_1(X_1, X_2)$ and $Y_2 = u_2(X_1, X_2)$ are 1-1 transforms

Let $y_1 = u_1(x_1, x_2)$ and $y_2 = u_2(x_1, x_2)$ have inverse functions $x_1 = w_1(y_1, y_2)$ and $x_2 = w_2(y_1, y_2)$

Then $g(y_1, y_2) = f[w_1(y_1, y_2), w_2(y_1, y_2)]$

Continuous

As above, but now $g(y) = f[w(y)]\,|J|$ where $J = w'(y)$ is the Jacobian of the transform

Also as above, but now $g(y_1, y_2) = f[w_1(y_1, y_2), w_2(y_1, y_2)]\,|J|$

where J is given by the 2 by 2 determinant: $J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix}$

Note: These concepts basically follow the rules for integral substitution or transforms in two variables

Moments

r^{th} moment about the origin $\mu_r' = E(X^r) = \begin{cases} \sum_x x^r f(x) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} x^r f(x)\,dx & \text{if } X \text{ is continuous} \end{cases}$

moments can also be taken about the mean – for example the variance is the 2^{nd} moment about the mean

moment generating functions and their properties

$$M_X(t) = E(e^{tx}) = \begin{cases} \sum_x e^{tx} f(x) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} e^{tx} f(x)\,dx & \text{if } X \text{ is continuous} \end{cases} \qquad \frac{d^r M_X(t)}{dt^r} = \mu_r'$$

If two r.v.'s have the same moment generating functions, then they have the same probability dist.

$M_{X+a}(t) = e^{at} M_X(t)$

If $S_n = \sum_{i=1}^{n} a_i X_i$ then $M_{S_n}(t) = M_{X_1}(a_1 t) M_{X_2}(a_2 t) \cdots M_{X_n}(a_n t)$

Discrete Distribution	Probability Function	Moment Generating Function
Binomial	$f(x)=\binom{n}{x}p^x(1-p)^{n-x}$, $x=0,1,\ldots,n$	$\left[pe^t+(1-p)\right]^n$
Geometric	$f(x) = p(1-p)^{x-1}$, $x = 1, 2, \ldots$	$\dfrac{pe^t}{1-(1-p)e^t}$
Poisson	$f(x)=\dfrac{\lambda^x e^{-\lambda}}{x!}$, $x=0, 1, \ldots$	$\exp\left[\lambda\left(e^t-1\right)\right]$

Continuous Distribution	Probability Function	Moment Generating Function
Uniform	$f(x)=\dfrac{1}{b-a}$, $a \le x \le b$	$\dfrac{e^{bt}-e^{at}}{t(b-a)}$
Normal	$f(x)=\dfrac{1}{\sigma\sqrt{2\pi}}\exp\left[-\dfrac{(x-\mu)^2}{2\sigma^2}\right]$, $-\infty < x < \infty$	$\exp\left(\mu t+\dfrac{t^2\sigma^2}{2}\right)$
Exponential	$f(x)=\dfrac{1}{\beta}e^{-x/\beta}$, $\beta>0$, $0\le x<\infty$	$(1-\beta t)^{-1}$
Gamma	$f(x)=\dfrac{x^{\alpha-1}e^{-x/\beta}}{\Gamma(\alpha)\beta^{\alpha}}$, $0<x<\infty$	$(1-\beta t)^{-\alpha}$
Chi-Square	$f(x)=\dfrac{x^{(\nu/2)-1}e^{-x/2}}{\Gamma(\nu/2)2^{\nu/2}}$, $0<x<\infty$	$(1-2t)^{-\nu/2}$

The moment generating functions for hypergeometric, beta, and lognormal distributions either do not exist or are too complicated to express in closed form.