BINOMIAL vs. POISSON vs. NORMAL DISTRIBUTIONS

Rule of thumb:

- Use Poisson to approximate Binomial when n is large and p is small. Let $\lambda = np$.
- Use Normal to approximate Binomial when both $np > 5$ and $nq > 5$. Let $\mu = np$, $\sigma^2 = npq$

Limiting Form of the Binomial is the Poisson distribution

$$\lim_{n\to\infty} \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} = \frac{\lambda^k e^{-\lambda}}{k!} \text{ where } \lambda = np$$

Proof:

$$\frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} = \frac{n(n-1)\cdots(n-k+1)}{k!}\left(\frac{\lambda}{n}\right)^k \left(1-\frac{\lambda}{n}\right)^{n-k}$$

$$= \frac{n(n-1)\cdots(n-k+1)}{n^k}\frac{\lambda^k}{k!}\left(1-\frac{\lambda}{n}\right)^{n-k}$$

$$= 1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)\frac{\lambda^k}{k!}\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-k}$$

As $n \to \infty$ while k and λ remain constants,

$$\lim_{n\to\infty} 1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right) = 1 \quad \text{and} \quad \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{-k} = 1$$

and using L'Hôpital's Rule

$$\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n = e^{-\lambda}$$

Then, using all the limits from above $b(k; n, p) \to 1\frac{\lambda^k}{k!}e^{-\lambda}1 = \frac{\lambda^k e^{-\lambda}}{k!}$

Note: Let λ = number of customers arriving per time unit. Then, if that time unit is broken up into n smaller sub-intervals, the probability of an arrival in that sub-interval is λ/n. The probability of k arrivals is given by $\frac{n!}{k!(n-k)!}\left(\frac{\lambda}{n}\right)^k\left(1-\frac{\lambda}{n}\right)^{n-k}$ and the limit is the Poisson distribution.

Asymptotic Expansion of the Tail of the Normal Distribution

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-t^2/2} dt$$

$$\left\{ \begin{array}{l} \text{let } u = \frac{1}{t},\ dv = te^{-t^2/2}dt \\ \text{then } du = -\frac{1}{t^2}dt,\ v = -e^{-t^2/2} \end{array} \right\}$$

$$Q(x) = \frac{1}{\sqrt{2\pi}} \left[\left(-\frac{1}{t} e^{-t^2/2} \right) \Bigg|_x^{\infty} - \int_x^{\infty} \frac{1}{t^2} e^{-t^2/2} dt \right]$$

$$Q(x) = \frac{1}{\sqrt{2\pi}} \left(\frac{e^{-x^2/2}}{x} \right) - \frac{1}{\sqrt{2\pi}} \int_x^{\infty} \frac{1}{t^2} e^{-t^2/2} dt$$

$$\left\{ \begin{array}{l} \text{let } u = \frac{1}{t^3},\ dv = te^{-t^2/2}dt \\ \text{then } du = -\frac{3}{t^4}dt,\ v = -e^{-t^2/2} \end{array} \right\}$$

$$Q(x) = \frac{1}{\sqrt{2\pi}} \frac{e^{-x^2/2}}{x} - \frac{1}{\sqrt{2\pi}} \left[\left(-\frac{1}{t^3} e^{-t^2/2} \right) \Bigg|_x^{\infty} - \int_x^{\infty} \frac{3}{t^4} e^{-t^2/2} dt \right]$$

$$Q(x) = \frac{1}{\sqrt{2\pi}} \frac{e^{-x^2/2}}{x} - \frac{1}{\sqrt{2\pi}} \frac{e^{-x^2/2}}{x^3} + \frac{1}{\sqrt{2\pi}} \int_x^{\infty} \frac{3}{t^4} e^{-t^2/2} dt$$

Continuing, we find

$$Q(x) = \frac{e^{-x^2/2}}{x\sqrt{2\pi}} \left(1 - \frac{1}{x^2} + \frac{1 \bullet 3}{x^4} - \frac{1 \bullet 3 \bullet 5}{x^6} + \cdots \right)$$

For example,

$Q(6) = 9.86 \text{ x } 10^{-10}$
$Q(10) = 7.62 \text{ x } 10^{-24}$
$Q(20) = 2.75 \text{ x } 10^{-89}$
$Q(30) = 4.91 \text{ x } 10^{-198}$

$z_{0.0001} = z_{10^{-4}} = 3.71902$
$z_{10^{-5}} = 4.265$
$z_{10^{-6}} = 4.753$
$z_{10^{-10}} = 6.361$
$z_{10^{-20}} = 9.262$