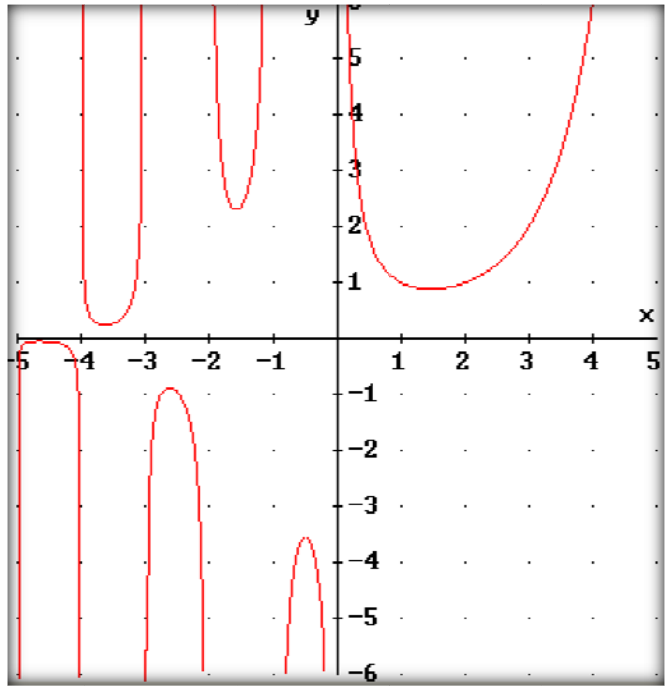
Gamma & Beta Function – Prof. Richard B. Goldstein

Gamma Function



$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt$$

Properties:

$$\Gamma(z) = (z-1)\Gamma(z-1) \quad \forall z \text{ real or complex}$$

$$\Gamma(n+1) = n! \quad n \text{ an integer}$$

$$\Gamma(n) = (n-1)!$$

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

Stirling's Approximations:

$$\Gamma(z) \sim e^{-z} z^{z-\frac{1}{2}} \sqrt{2\pi}\left(1 + \frac{1}{12z} + \frac{1}{288z^2} - \frac{139}{51840z^3} - \frac{571}{2488320z^4} + \cdots\right)$$

$$\ln\Gamma(z) \sim \frac{1}{2}\ln(2\pi) + \left(z - \frac{1}{2}\right)\ln z - z + \frac{1}{12z} - \frac{1}{360z^3} + \frac{1}{1260z^5} - \cdots$$

Beta Function

$$B(x,y) = B(y,x) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \frac{(x-1)!(y-1)!}{(x+y-1)!}$$ where the last expression is for integer x and y

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt = 2\int_0^{\pi/2} \cos^{2x-1}\theta\, \sin^{2y-1}\theta\, d\theta$$

Incomplete Beta Function: $I_x(a,b) = \dfrac{B(x;a,b)}{B(a,b)}$ where $B(x;a,b) = \int_0^x t^{a-1}(1-t)^{b-1}\, dt$