Saving for and Spending in Retirement

Input Variables:

S	current salary ($)
p	percentage saved each year (%)
r	growth rate of investments (%)
g	growth rate of salary/raise (%)
C	current amount saved for retirement ($)
t	years until retirement (yrs)
L	life expectancy in retirement (yrs)

Output Variables:

FS	future salary ($)
FV	future value of savings ($)
FC	future value of current amount saved ($)
FT	future total in retirement ($)
YI	yearly income for first year of retirement ($)
q	percentage of yearly income needed in retirement w/o Social Security (%)
x	percentage to withdraw to make retirement last L years

Assumptions: Investments grow at r% each year and salary grows at g% until retirement in t years. The amount invested for retirement is proportional, being p% of each year's salary. Yearly income from investments continues to grow at g% in retirement and the amount invested continues to grow at r%. All calculations use continuous compounding, which means that the number of compounding periods, k, goes to infinity.

$\boxed{FS = Se^{gt}}$ is the future salary at retirement t years into the future

ACCUMULATION PHASE

$$FV = \frac{pS}{k}\left(1+\frac{r}{k}\right)^{kt-1} + \frac{pS}{k}\left(1+\frac{g}{k}\right)\left(1+\frac{r}{k}\right)^{kt-2} + \cdots + \frac{pS}{k}\left(1+\frac{g}{k}\right)^{kt-2}\left(1+\frac{r}{k}\right) + \frac{pS}{k}\left(1+\frac{g}{k}\right)^{kt-1}$$

$$FV = \frac{pS}{k}\frac{\left(1+\frac{r}{k}\right)^{kt} - \left(1+\frac{g}{k}\right)^{kt}}{\left(1+\frac{r}{k}\right) - \left(1+\frac{g}{k}\right)} = \frac{pS}{r-g}\left[\left(1+\frac{r}{k}\right)^{kt} - \left(1+\frac{g}{k}\right)^{kt}\right]$$

$$\lim_{k\to\infty} FV = \frac{pS}{r-g}\left(e^{rt} - e^{gt}\right)$$

$$\lim_{r\to g} FV = pSte^{gt}$$

$$\boxed{FV = \begin{cases} \frac{pS}{r-g}\left(e^{rt} - e^{gt}\right) & r \neq g \\ ptSe^{gt} & r = g \end{cases}}$$ is the future value of savings from future investments

The current amount saved for retirement grows from \$C to $\boxed{FC = Ce^{rt}}$

The total amount t years later at retirement has accumulated to FT = FV + FC.

$$\boxed{FT = \begin{cases} \dfrac{pS}{r-g}\left(e^{rt} - e^{gt}\right) + Ce^{rt} & r \neq g \\ \left(ptS + C\right)e^{rt} & r = g \end{cases}}$$

This represents a more realistic estimate of the total in retirement because not only do investments grow, but the amount added each year to retirement investments grows with one's salary.

Example: One has \$50,000 (C) saved for retirement and an annual salary of \$40,000 (S). Their salary grows at 3% (g) annually and 15% (p) of their salary is saved each year for retirement. If their investment grows at 5% (r), then the amount accumulated 30 years later when they retire is given by:

$$FT = \frac{pS}{r-g}\left(e^{rt} - e^{gt}\right) + Ce^{rt} = \frac{0.15(40000)}{0.05-0.03}\left(e^{0.05(30)} - e^{0.03(30)}\right) + 50000e^{0.05(30)}$$

$$FT = \frac{6000}{0.02}\left(4.481689... - 2.459603...\right) + 50000(4.481689...) = 606{,}625.79 + 224{,}084.45 = 830{,}710.24$$

The current amount saved of \$50,000 grows to \$224,085 and the amount added grows to \$606,626 for a total of almost \$830,710 at retirement.

SPENDING IN RETIREMENT

Now, that the savings nest egg has been calculated, how much can one spend in retirement to last L years, one's life expectancy? Again assume that the yearly income (YI) continues to grow at g% and investments grow at r% with continuous compounding. Let the present value, PV, be set to the nest egg, FT. That is PV = FT. What is the yearly income (YI) that one can be extract from their retirement account to last the L years of one's life expectancy?

$$PV = \frac{YI}{k} + \frac{YI}{k}\left(\frac{1+\frac{g}{k}}{1+\frac{r}{k}}\right) + \frac{YI}{k}\left(\frac{1+\frac{g}{k}}{1+\frac{r}{k}}\right)^2 + \frac{YI}{k}\left(\frac{1+\frac{g}{k}}{1+\frac{r}{k}}\right)^3 + \cdots + \frac{YI}{k}\left(\frac{1+\frac{g}{k}}{1+\frac{r}{k}}\right)^{kL-1}$$

$$PV = \frac{YI}{k}\left[\frac{1-\left(\frac{1+\frac{g}{k}}{1+\frac{r}{k}}\right)^{kL}}{1-\frac{1+\frac{g}{k}}{1+\frac{r}{k}}}\right]$$

$$\lim_{k\to\infty} PV = \frac{YI}{r-g}\left(1 - e^{(g-r)L}\right)$$

If g = r, then $PV = \frac{YI}{k}(kL) = YI * L$.

$$YI = \begin{cases} \dfrac{(r-g)PV}{1-e^{(g-r)L}} & \text{for } r \neq g \\ \dfrac{PV}{L} & \text{for } r = g \end{cases}$$

Example continued:

$$YI = \frac{(0.05-0.03)830710.24}{1-e^{(0.05-0.03)20}} = 0.0606649...(830710.24) = \$50,394.95$$

That is, if one can expect to live 20 (L) years, then the yearly income starts out a $50,394.95 and grows at the rate of 3% a year in retirement to $91,825.59 after 20 years. Because the final earned salary is $Se^{gt} = 40000e^{0.03(30)}$ =$98,384.12 this represents 51.22% of previous yearly income. Even with Social Security one may still feel short of the amount needed. What percentage, p, should be saved each year to achieve a yearly income (YI) that is q (%) of the final earned yearly income?

Setting PV =FT, solve YI = q*FS. This becomes,

$$\left[\frac{r-g}{1-e^{(g-r)L}}\right]\left[pS\left(\frac{e^{rt}-e^{gt}}{r-g}+Ce^{rt}\right)\right] = qSe^{gt}$$

Solving for p, $$p = \frac{qSe^{gt} - qSe^{gt+gL-rL} - C(r-g)e^{rt}}{S(e^{rt}-e^{gt})}$$

Example

If q = 0.6,

$$p = \frac{0.6(40000)e^{0.03(30)} - 0.6(40000)e^{0.03(30)+0.03(20)-0.05(20)} - 50000(0.05-0.03)e^{0.05(30)}}{40000(e^{0.05(30)} - e^{0.03(30)})}$$

$$p = \frac{59030.47 - 39569.32 - 4481.69}{80883.44} = 0.185198...$$

By increasing the proportion saved from 15% to 18.5% one can achieve 60% of one's previous salary and continue to have that yearly income grow at the 3% during retirement.

Suppose that both growth rates are equal, r = g = 3%. Then the future nest egg grows to

$FT = (ptS+C)e^{rt} = [(0.15)(30)(40000)+50000]e^{0.03(30)} = 230000(2.459603\ldots) = \$565,708.72$

YI = PV/L = 565708.72/20 = $28,285.44 which is only 28.75% of the previous income. Set $(ptS+C)e^{rt} = qSe^{rt}*L$ and solve for p = (qSL – C)/tS = (0.6*40000*20 – 50000)/(30*40000) = 0.35833… One would need to save 35.83% of one's salary each year if the rate of growth of salary and investments are equal. In this situation the actual value of r and g are not needed.

How long can one expect their retirement savings to last with a yearly withdrawal (W) that grows each year at the rate of g%?

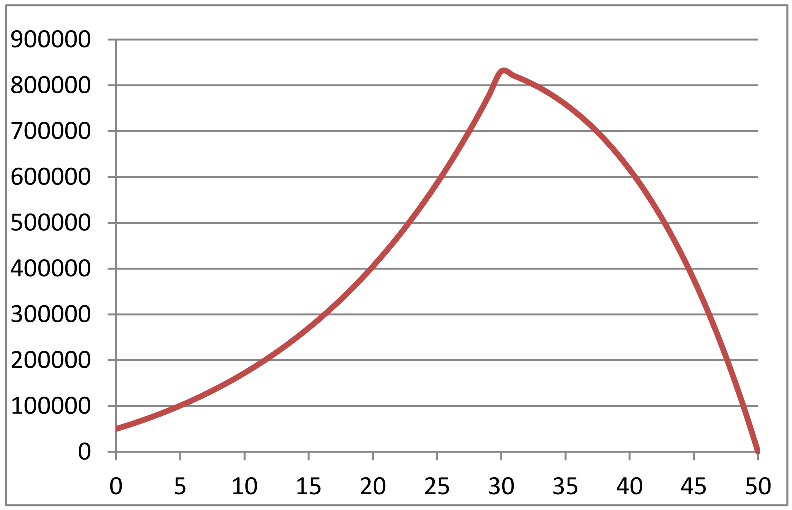
Using $W = \begin{cases} \dfrac{(r-g)PV}{1-e^{(g-r)L}} & \text{for } r \neq g \\ \dfrac{PV}{L} & \text{for } r = g \end{cases}$ and solving for L we find $L = \begin{cases} \ln\left(1-\dfrac{(r-g)PV}{W}\right)/(g-r) & g \neq r \\ PV/W & g = r \end{cases}$

If W = $59,030.47 (60% of final salary), then L = 14.22 years, short of the 20 year life expectancy. If the funds grow at 3% along with the needs, then the money saved will only last 9.58 years and if C = 0 it will only last L = p*S*t/(q*S) = pt/q = 0.15(30)/0.6 = 7.5 years again where the values of r and g are not needed.

BALANCE AFTER t YEARS

$$B(t) = \begin{cases} PVe^{rt} - W\left(\dfrac{e^{rt}-e^{gt}}{r-g}\right) & r \neq g \\ (PV - Wt)e^{rt} & r = g \end{cases}$$

The graph shows the accumulation while working and spending in retirement. One starts with $50,000, peaks at $830,710 after 30 years, and finally runs out at the end of 20 years.



ALTERNATIVE

Perhaps the assumption of a yearly income growth equal to previous salary growth rates is too demanding and the years of income is needed might exceed life expectancy.

Example

Using $YI = \dfrac{(r-g)PV}{1-e^{(g-r)L}}$ let us assume PV = $1,000,000, r = 5%, g = 1.5%, and L = 25.

$$YI = \frac{(0.05-0.015)1000000}{1-e^{(0.015-0.050)25}} = \frac{35000}{1-0.416862...} = \$60{,}020.10 \text{ (this is 6.00\%)}$$

If one can live with a 1.5% yearly increase (the current proposed Social Security increase for 2014) and can achieve an average return of 5% on investments, then one can have their nest egg last 25 years and they can withdraw 6% of that nest egg starting the first year.

If one only withdraws $54,000 to start, then the $1,000,000 can last longer:

$$L = \frac{\ln\left(1-\dfrac{(0.050-0.015)1000000}{54000}\right)}{0.015-0.050} = \frac{-1.044545...}{-0.035} = 29.84 \text{ years}$$

APPENDIX

$$\frac{a^n - b^n}{a-b} = a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + ab^{n-2} + b^{n-1}$$

$\lim_{k\to\infty}\left(1+\frac{r}{k}\right)^{kt} = e^{rt}$ comes from the use of L'Hôpital's rule where x = 1/k and take exponential function:

$$\lim_{k\to\infty} \ln\left(1+\frac{r}{k}\right)^{kt} = \lim_{k\to\infty} kt \ln\left(1+\frac{r}{k}\right) = \lim_{x\to 0}\frac{t\ln(1+rx)}{x} \Rightarrow \lim_{x\to 0}\frac{\frac{rt}{1+rx}}{1} = rt$$