Does the Existence of First Partial Derivatives imply Continuity?

1-D **Yes** (see sect 2.8 page 158 in Stewart)

2-D & higher **No** (see counterexample below)

$$\text{Let } f(x) = \begin{cases} \dfrac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0,0) \\ 0 & \text{if } (x, y) = (0,0) \end{cases}$$

Continuity

Letting y = mx we get f(x, y) = f(x, mx) = $\dfrac{mx^2}{x^2 + m^2x^2} = \dfrac{m}{1+m^2}$. Therefore on each approach to (0, 0) other than the two axes we do not approach 0.

First Partial Derivatives

$$\frac{\partial f}{\partial x} = f_x(x, y) = \frac{(x^2 + y^2)y - xy(2x)}{(x^2 + y^2)^2} = \frac{y^3 - x^2y}{(x^2 + y^2)^2} \text{ for all } (x, y) \neq (0,0)$$

$$\text{At } (0, 0) \left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \lim_{h \to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{\frac{h(0)}{h^2 + 0^2} - 0}{h} = \lim_{h \to 0} \frac{0}{h} = 0.$$

The derivative exists everywhere (although it is not itself continuous)!

$$\text{Similarly, } \frac{\partial f}{\partial y} = f_y(x, y) = \frac{(x^2 + y^2)x - xy(2y)}{(x^2 + y^2)^2} = \frac{x^3 - xy^2}{(x^2 + y^2)^2} \text{ for all } (x, y) \neq (0,0)$$

$$\text{At } (0, 0) \left.\frac{\partial f}{\partial y}\right|_{(0,0)} = \lim_{k \to 0} \frac{f(0,0+k) - f(0,0)}{k} = \lim_{k \to 0} \frac{\frac{0(k)}{0^2 + k^2} - 0}{k} = \lim_{k \to 0} \frac{0}{k} = 0.$$

This partial derivative also exists everywhere. We have both first partial derivatives existing everywhere, yet the function is not continuous at the origin.

Conclusion

The existence of first partial derivatives at a point does not imply continuity at that point in two (or higher) dimensions. This function is not differentiable (see page 895 section 14.4 for a definition) because the first partial derivatives are not continuous.