Counterexample for Mixed Partial Derivative

For most textbook examples in multivariate calculus $f_{xy}(a, b) = f_{yx}(a, b)$. The following counterexample shows a function where f_x and f_y are continuous everywhere and yet $f_{xy}(0, 0) \neq f_{yx}(0, 0)$ because they are not continuous functions at the origin. In fact, $f_{xy}(0, 0) = 1$ and $f_{yx}(0, 0) = -1$.

Clairaut's Theorem Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then $f_{xy}(a, b) = f_{yx}(a, b)$.

$$f(x,y) = \begin{cases} xy\dfrac{x^2 - y^2}{x^2 + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

f(x, y) is continuous

Using polar coordinates, $x = r\cos\theta$ and $y = r\sin\theta$, f(x, y) = $f(r,\theta) = r^2\cos\theta\sin\theta\dfrac{r^2(\cos^2\theta - \sin^2\theta)}{r^2(\cos^2\theta + \sin^2\theta)} = \dfrac{r^2\sin 4\theta}{4}$

Therefore $|f(xy)| \leq r^2/4 \to 0$ as $r \to 0$, and f(x, y) is continuous at the origin.

f_x(x, y) and f_y(x, y) are both continuous

$$|f_x(x,y)| = \left|\frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2}\right| = \left|r(\cos^4\theta\sin\theta + 4\cos^2\theta\sin^3\theta - \sin^5\theta)\right| \leq 6r \to 0 \text{ as } r \to 0$$

$$f_x(0,0) = \lim_{h\to 0}\frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0}\frac{(h)(0)\dfrac{h^2-0}{h^2+0} - 0}{h} = \lim_{h\to 0}\frac{0}{h} = 0$$

$$|f_y(x,y)| = \left|\frac{x^5 - 4x^3y^2 - xy^4}{(x^2+y^2)^2}\right| = \left|r(\cos^5\theta - 4\cos^3\theta\sin^2\theta - \cos\theta\sin^4\theta)\right| \leq 6r \to 0 \text{ as } r \to 0$$

$$f_y(0,0) = \lim_{k\to 0}\frac{f(0,0+k) - f(0,0)}{k} = \lim_{k\to 0}\frac{(0)(k)\dfrac{0-k^2}{0+k^2} - 0}{k} = \lim_{k\to 0}\frac{0}{k} = 0$$

f_{xy}(x,y) and f_{yx}(x,y) are not continuous at the origin

$$f_{xy}(x,y) = f_{yx}(x,y) = \frac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{(x^2+y^2)^3} = \cos^6\theta + 9\cos^4\theta\sin^2\theta - 9\cos^2\theta\sin^4\theta - \sin^6\theta = \cos 2\theta(1 + 2\sin^2 2\theta)$$

If θ = 0 (x-axis), then $f_{xy}(x,y) = 1$. If θ = π/2 (y-axis), then $f_{xy}(x,y) = -1$. $f_{xy}(x, y)$ is dependent on θ only.

$$f_{xy}(0,0) = \lim_{k\to 0}\frac{f_x(0,0+k) - f_x(0,0)}{k} = \lim_{k\to 0}\frac{\dfrac{-k^5}{k^4} - 0}{k} = \lim_{k\to 0}(-1) = -1$$

$$f_{yx}(0,0) = \lim_{h\to 0}\frac{f_y(0+h,0) - f_y(0,0)}{h} = \lim_{h\to 0}\frac{\dfrac{h^5}{h^4} - 0}{h} = \lim_{h\to 0}(+1) = 1$$