ADVANCED TOPICS OF DIFFERENTIATION

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CHAIN RULE

If y = f(u) and u = g(x), define the composite function y = h(x) = f[g(x)].

Then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ provided that $\frac{dy}{du}$ and $\frac{du}{dx}$ exist.

Or, equivalently, $h'(x) = f'[g(x)] \bullet g'(x)$

$$\boxed{\begin{aligned} &\frac{d}{dx}\left[u(x)\right]^n = n\left[u(x)\right]^{n-1}u'(x) \\ &\frac{d}{dx}\ln\left[u(x)\right] = \frac{1}{u(x)}u'(x) = \frac{u'(x)}{u(x)} \\ &\frac{d}{dx}e^{u(x)} = e^{u(x)}u'(x) \end{aligned}}$$

<u>Examples</u>

$$\frac{d}{dx}\left(x^3 - 5x + 1\right)^5 = 5\left(x^3 - 5x + 1\right)^4 \bullet (3x^2 - 5) = (15x^2 - 25)(x^3 - 5x + 1)^4$$

$$\frac{d}{dx}\sqrt{x^3 - 5x + 1} = \frac{d}{dx}\left(x^3 - 5x + 1\right)^{1/2} = \tfrac{1}{2}\left(x^3 - 5x + 1\right)^{-1/2} \bullet (3x^2 - 5) = \frac{3x^2 - 5}{2\sqrt{x^3 - 5x + 1}}$$

$$\frac{d}{dx}\ln(x^3 - 5x + 1) = \frac{3x^2 - 5}{x^3 - 5x + 1}$$

$$\frac{d}{dx}e^{x^3 - 5x + 1} = e^{x^3 - 5x + 1}(3x^2 - 5) = (3x^2 - 5)e^{x^3 - 5x + 1}$$

IMPLICIT DIFFERENTIATION

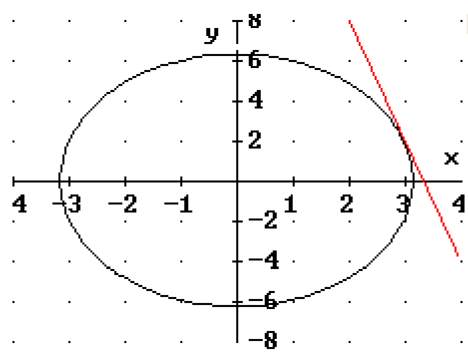
Think of y as y(x). That is, since y is a function of x, but it is not explicitly given, we simply write $\frac{d}{dx}y(x) = y'$ to indicate its derivative.

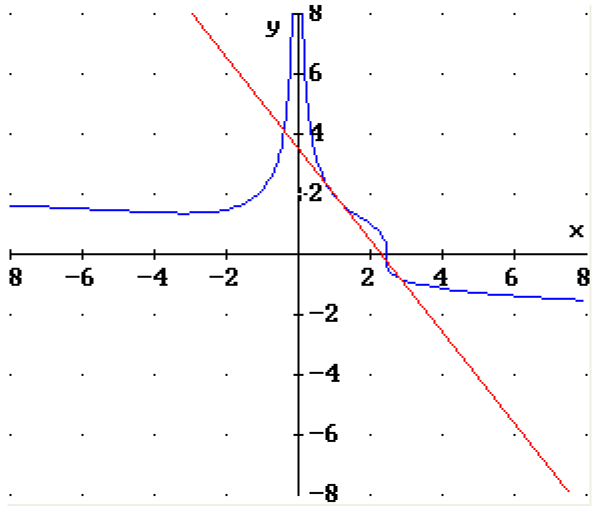
<u>Examples</u>

$$\frac{d}{dx}y^n = ny^{n-1}y' \qquad \frac{d}{dx}y^3 = 3y^2y'$$

$$4x^2 + y^2 = 40 \Rightarrow 8x + 2yy' = 0 \Rightarrow y' = -\frac{8x}{2y} = -\frac{4x}{y}$$

For example at (3,2) slope $m = -\frac{12}{2} = -6$





$$\frac{d}{dx}2x^2y^3 = [4x]\bullet\left[y^3\right]+\left[2x^2\right]\bullet\left[3y^2y'\right] = 4xy^3 + 6x^2y^2y'$$

$$x^3 + 2x^2y^3 - y = 15 \text{ at } (1,2)$$

$$3x^2 + 4xy^3 + 6x^2y^2y' - y' = 0 \Rightarrow\ y' = -\left.\frac{3x^2+4xy^3}{6x^2y^2-1}\right|_{(1,2)} = -\frac{3+32}{24-1} = -\frac{35}{23}$$

RELATED RATES

Think of x and y (or any other variables) as functions of t. Then,

$$\frac{d}{dt}x(t) = \dot{x},\ \frac{d}{dt}y(t) = \dot{y}, \text{ and } \frac{d}{dt}x^3 = 3x^2\dot{x}$$

<u>Examples</u>

$$V = \frac{4}{3}\pi r^3 \Rightarrow\ \dot{V} = \frac{4}{3}\pi(3r^2\dot{r}) = 4\pi r^2\dot{r}$$

For example, if the radius of a balloon is 3 cm and it is growing at 1 cm/sec then its volume is growing at $4\pi(3^2)(1) = 36\pi\ \text{cm}^3/\text{sec}$

$$P(x) = -100 + 20x - 0.1x^2 \Rightarrow\ \dot{P} = (20 - 0.2x)\dot{x}$$

For example if x = 20 the profit P(20) = -100 + 400 – 40 = 260. If x is increasing by 3 items per week, $\dot{x} = 3$ and $\dot{P} = (20 - 0.2(20))\bullet 3 = 16\bullet 3 = \$48/\text{week}$ Note that P(23)-P(20) = \$307.10 - \$260.00 = \$47.10.